Constructing the Rational Numbers

Recall: definition

A rational number is a quotient (ratio) of numbers m and n with $m, n \in \mathbb{Z}$, $n \neq 0$:

$$\frac{m}{n} \in \mathbb{Q} = \text{rational numbers.}$$

Equivalence Relation

Consider

$$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

where "$\mathbb{Z} \setminus \{0\}$" means "the integers without zero."

Define $(m,n) \sim (k,\ell)$

$$\boxed{m\ell = nk}$$

Why is this well-defined?

1) $(a,b) \sim (a,b)$

$ab = ab$ ✓

2) If $(a,b) \sim (m,n)$, then $(m,n) \sim (a,b)$

$(a,b) \sim (m,n)$ means

$an = bm$, so

$na = mb$

$\Rightarrow (m,n) \sim (a,b)$ ✓

3) If $(a,b) \sim (m,n)$ and $(m,n) \sim (k,\ell)$, then $(a,b) \sim (k,\ell)$

Suppose $(a,b) \sim (m,n)$ and $(m,n) \sim (k,\ell)$. Then

$an = bm$ and $m\ell = nk$.

Much like for the construction of the integers, we can now define a limited notion of divisibility.

If $s, t \in \mathbb{Z}$ and $t \neq 0$, define

$\frac{s}{t} \in \mathbb{Z}$ as the number

$c \in \mathbb{Z}$ with $c \cdot t = s$.

If no such number exists in $\mathbb{Z}$, division is not defined.

Hence, $m = \frac{an}{b}$ is

well-defined since

$bm = an$.

Substituting $m = \frac{an}{b}$

into $m\ell = nk$,

$$\frac{an}{b} \cdot \ell = nk$$

Multiplying by b and dividing by n, we obtain

$$a\ell = bk$$

$$\Rightarrow (a,b) \sim (k,\ell)$$

We regard the integers as

$$n \in \mathbb{Z} \longleftrightarrow [(n,1)] \in \frac{\mathbb{Z} \times \mathbb{Z}\setminus\{0\}}{\sim}$$

The $0 = [(0,1)]$.

Note that $(0,1) \sim (0,n)$

$\forall \; n \in \mathbb{Z}\setminus\{0\}$.

Positivity and Negativity

Claim: If $[(m,n)] \in \mathbb{Z}\setminus\{0\} \times \mathbb{Z}\setminus\{0\} / \sim$

then $\exists\ [(a,b)] \in \mathbb{Z} \times \mathbb{N}$

with $[(m,n)] = [(a,b)]$

If $n > 0$, there is nothing to prove, so suppose $n < 0$.

Then $m(-n) = -nm$
$= n(-m)$

Hence with $a = -m$, $b = -n$,

$$[(a,b)] = [(m,n)].$$

The positive elements will then be $[(a,b)]$ with $a, b \in \mathbb{N}$.

The negative elements will be $[(a,b)]$ with $a < 0$, $b \in \mathbb{N}$.

You can check that this respects the notions of positivity and negativity from $\mathbb{Z}$.

Addition and Multiplication

Unlike the construction of $\mathbb{Z}$ from $\mathbb{N}$, here multiplication is straightforward:

$$\boxed{[(a,b)\cdot(m,n)] = [(am,bn)]}$$

Addition has to reflect the need for a common denominator when adding fractions. Hence,

$$\boxed{[(a,b)] + [(m,n)] = [(an+mb,\ bn)]}$$

You can check that all the desired properties (commutativity, associativity, etc.) hold.

Division

From HW: Every nonzero element in $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z} \setminus \{0\} / \sim$ has a multiplicative inverse.

We then define, for $m \neq 0$,

$$\boxed{\frac{[(a,b)]}{[(m,n)]} = [(a,b)] \cdot \left([(m,n)]\right)^{-1}}$$

This is compatible with our earlier notion of divisibility in $\mathbb{Z}$.

Ordering

Define "$<$" on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\} / \sim$

by

$$\boxed{[(a,b)] < [(m,n)] \text{ if } an < bm}$$

Again, you can check that all familiar properties (positive greater than negative, etc)